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Singular Measures on the Unit Circle and Their Reflection Coefficients

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Orthogonal polynomials on the unit circle are determined by their reflection coefficients through the Szegő recurrences. In the present paper we examine two particular classes of measures on the unit circle. The first one consists of measures whose reflection coefficients tend to the unit circle. For such measures we give complete description of their supports (up to the set of isolated masspoints) in terms of reflection coefficients. The supports of measures from the second class have finitely many limit points. We prove the unit circle analogue of M. G. Krein's characterization for the similar class of measures on the real line. The examples of measures from both classes are given. © 1999 Academic Press

Key Words: unit circle orthogonal polynomials; reflection coefficients; perturbation theory; spectral mapping.

1. INTRODUCTION

Orthonormal polynomials on the unit circle $\mathbb{T} =^{\text{def}} \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ are defined by

$$\int_{\mathbb{T}} \varphi_n(\mu, \zeta)\, \overline{\varphi_m(\mu, \zeta)}\, d\mu = \delta_{m,n}, \qquad m, n \in \mathbb{Z}^+ \stackrel{\text{def}}{=} \{0, 1, 2, ...\},$$

where μ is a probability measure $\mathbb{T}$ with infinite support, supp(μ) (which is, by definition, the smallest closed set whose complement has μ measure 0), and

$$\varphi_n(\mu, z) = \kappa_n(\mu) z^n + \text{lower degree terms}, \qquad \kappa_n(\mu) > 0.$$

The monic orthogonal polynomials are $\Phi_n(\mu) = \kappa_n^{-1} \varphi_n(\mu)$. The *reflection coefficients* $a_n(\mu) =^{\text{def}} \Phi_n(\mu, 0)$ describe completely not only the monic orthogonal polynomials (through the Szegő recurrences) but the orthonormal polynomials as well.

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There is one-to-one corespondence between probability measures on $\mathbb{T}$ and sequences of complex numbers $\{a_n\}$ with $|a_n|<1$. According to the analogue of Favard's theorem for the unit circle for each such sequence there exists a unique probability measure μ on $\mathbb{T}$ with $\Phi_n(\mu, 0)=a_n$ (see, e.g. [10] for a simple proof of this result). Thereby, we arrive at apparently quite natural (although rather intricate) parametization of the set of all probability measures on $\mathbb{T}$.

The relation between the properties of measures and the asymptotic behavior of their reflection coefficients has been extensively studied lately. The starting point for our investigation is the following result (cf. [14, Theorem 6]).

THEOREM A. *Let μ be a probability measure on $\mathbb{T}$ having an infinite support, and let $\tau\in\mathbb{T}$. Then the following statements are equivalent.*

(1) *The derived set $\{\text{supp}(\mu)\}'$ of the support of μ is equal to $\{\tau\}$.*

(2) *We have $\lim_{n\to\infty}\Phi_{n+1}(\mu,0)\,\overline{\Phi_n(\mu,0)}=-\tau$.*

(3) *We have $\lim_{n\to\infty}\int_{\mathbb{T}}(\zeta-\tau)\,\varphi_n(\mu,\zeta)\,\overline{\varphi_{n+k}(\mu,\zeta)}\,d\mu=0$ for all $k\in\mathbb{Z}$.*

We are going to extend this theorem in two directions. The first one is motivated by the result due to D. Maki and T. Chihara (see [5], [19]).

THEOREM B. *Let σ be a nonnegative Borel measure on the real line with infinite support whose moments are all finite. Let $\{p_n\}$ be a system of orthonormal polynomials with respect to σ which satisfies the three-term recurrence relation*

$$xp_n(x)=a_{n+1}p_{n+1}(x)+b_np_n(x)+a_np_{n-1}(x),\qquad a_n>0,\quad b_n=b_n.$$

Assume that $\lim_{n\to\infty}a_n=0$ and denote by $\mathscr{L}$ the set of limit points of the sequence $\{b_n\}$. Then

$$\{\text{supp}(\sigma)\}'=\mathscr{L}. \tag{1}$$

In Section 2 we study measures on $\mathbb{T}$ which can be viewed as the unit circle counterparts of those in Theorem B.

DEFINITION 1. A probability measure μ on $\mathbb{T}$ with infinite support belongs to class $\mathscr{S}$ if

$$\lim_{n\to\infty}|\Phi_n(\mu,0)|=1. \tag{2}$$

Our main result herein is the unit circle analogue of Theorem B for the class $\mathcal{S}$. An operator theoretic approach to the theory of orthogonal polynomials on the unit circle proves helpful now. The equivalence (1)–(2) in Theorem A emerges then as a very special case of Theorem 5 below. Note that, as it follows from [20, Lemma 4, p. 110], the measures $\mu \in \mathcal{S}$ are singular with respect to Lebesgue measure on $\mathbb{T}$.

The second direction is initiated by M. G. Krein's characterization of compactly supported nonnegative Borel measures on the real line whose support contains finitely many limit points (see [3, Theorem 2, p. 230]). We will state M. G. Krein's theorem in a slightly modified form (see [9, p. 152]).

THEOREM C. *Let σ be a nonnegative Borel measure on the real line with infinite support whose moments are all finite. Denote by A the multiplication operator $(Af)(x) \stackrel{\text{def}}{=} xf(x)$ in $L^2(\sigma, \mathbb{R})$. Let $x_1, x_2, ..., x_N$ be N distinct real points. Then the following statements are equivalent.*

(1) $\{\operatorname{supp}(\sigma)\}' = \{x_1, x_2, ..., x_N\}$.

(2) *For any polynomials p an operator $p(A)$ is compact in $L^2(\sigma, \mathbb{R})$ if and only if $p(x_j) = 0$ for $j = 1, 2, ..., N$.*

DEFINITION 2. Let $\tau_1, \tau_2, ..., \tau_N$ be N distinct points on $\mathbb{T}$. A probability measure μ on $\mathbb{T}$ with infinite support belongs to class $\mathcal{K}(\tau_1, ..., \tau_N)$ if

$$\{\operatorname{supp}(\mu)\}' = \{\tau_1, \tau_2, ..., \tau_N\}. \tag{3}$$

In Section 3 we prove the unit circle analogue of Theorem C for $\mu \in \mathcal{K}(\tau_1, ..., \tau_N)$. It turns out that the case $N = 2$ can be tackled explicitly in terms of the reflection coefficients. The latter situation is illustrated by the symmetrized Al-Salam–Carlitz polynomials for the unit circle (cf. [22, Section 6]).

2. DERIVED SET OF SUPPORT AND REFLECTION COEFFICIENTS

Let us begin with some definitions and basic results from the theory of linear operators in a Hilbert space (spectrum of a linear operator, structure of spectrum, H. Weyl's perturbation theorem). Among the variety of terminological conventions we adopt one from [13, Chapter 1, Sects. 1.1–1.5], which suits our purpose perfectly.

Let T be a bounded linear operator in a Hilbert space $\mathcal{H}$. Denote by $\operatorname{sp}(T)$ the spectrum of T, that is, the set of all those complex numbers λ for which the operator $T-\lambda I$ is not invertible. The spectrum of a linear operator is known to be a compact subset of the complex plane which is contained in the disk $\{\lambda\in\mathbb{C}: |\lambda|\leqslant \|T\|\}$.

If we analize the reasons for an operator to be not invertible we come up with the following classification of the specrum.

DEFINITION. A complex number λ belongs to a *discrete spectrum* $\mathrm{sp}_d(T)$, if there is a unit vector h for which $Th=\lambda h$. A complex number λ belongs to a *continuous spectrum* $\mathrm{sp}_c(T)$, if there is a noncompact sequence of unit vectors $\{h_n\}$ such that $\lim_{n\to\infty}(T-\lambda I)\,h_n=0$.

It is clear that

$$\mathrm{sp}_d(T)\cup\mathrm{sp}_c(T)\subset\mathrm{sp}(T), \tag{4}$$

wherein the possibility of a proper inclusion is not ruled out. Note that the intersection of two parts of the spectrum may be nonempty (see Example 3 below). It is well known that for *normal* operators (i.e., $T^*T=TT^*$) the equality holds in (4).

Let us illustrate the foregoing definition with two examples.

EXAMPLE 3. The simplest and yet by far the most important example of a normal operator is a *multiplication* operator. Let $\mathscr{F}$ be a compact set in the complex plane and ν be a non-negative finite Borel measure with $\mathrm{supp}(\nu)\subset\mathscr{F}$. In the Hilbert space $L^2(\nu,\mathscr{F})$ of measurable and square-integrable functions on $\mathscr{F}$ with the inner product and norm

$$\langle f,g\rangle_\nu \overset{\text{def}}{=} \int_{\mathscr{F}} f\bar{g}\,d\nu, \qquad \|f\|_\nu \overset{\text{def}}{=} \sqrt{\langle f,f\rangle_\nu},$$

consider the operator $(Af)(z)=zf(z)$. It is easy to see that $\lambda\in\mathrm{sp}_d(A)$ if and only if $\nu\{\lambda\}>0$. A bit more elaborate analysis shows that

$$\lambda\in\mathrm{sp}_c(A)\Leftrightarrow\lambda\in\{\mathrm{supp}(\nu)\}',$$

that is, λ belongs to the derived set of $\mathrm{supp}(\nu)$. In other words, the part of $\mathrm{supp}(\nu)$ in a punctured disk $B_\varepsilon(\lambda)=^{\text{def}}\{z: 0<|z-\lambda|<\varepsilon\}$ has positive ν-measure for each $\varepsilon>0$. Therefore the nonisolated masspoints in $\mathscr{F}$ constitute the intersection $\mathrm{sp}_d(A)\cap\mathrm{sp}_c(A)$.

EXAMPLE 4. Let $D=^{\text{def}}\mathrm{diag}\{z_0,z_1,...\}$ be a diagonal operator in the Hilbert space ℓ^2. Then $\mathrm{sp}_d(D)=\{z_n\}$ and $\mathrm{sp}_c(D)=\{z_n\}'$, that is, the set of all limit points of the set $\{z_n\}$.

A key role throughout the whole paper is played by H. Weyl's perturbation theorem. For reader's convenience we provide a brief proof of the simple version of this result we need (cf. [13, Theorem 18, p. 22]).

THEOREM (H. Weyl). *Let* T_1 *and* T_2 *be linear operators for which the difference* $T_2 - T_1$ *is a compact operator. Then* $\mathrm{sp}_c(T_1) = \mathrm{sp}_c(T_2)$.

Proof. We proceed in two steps.

Step 1. Let $\{f_n\}$ be a noncompact sequence of unit vectors. We show that there is a subsequence $\{f_n\}_{n \in \Lambda}$ for which the consecutive differences $\{f_{n+1} - f_n\}_{n \in \Lambda}$ form a noncompact sequence. The noncompactness of $\{f_n\}$ implies the lack of finite ε-net for some $\varepsilon > 0$. In other words, there is a subsequence $\{f_n\}_{n \in \Lambda'}$ with the property

$$\|f_n - f_m\| \geqslant \varepsilon, \qquad n, m \in \Lambda'.$$

It is well known, that each bounded sequence of vectors in a Hilbert space contains a weakly convergent subsequence. Let f_n converge weakly to some vector f for $n \in \Lambda \subset \Lambda'$. Then $f_{n+1} - f_n$ weakly converges to zero along Λ. On the other hand $\|f_{n+1} - f_n\| \geqslant \varepsilon$, $n \in \Lambda$, and thereby $\{f_{n+1} - f_n\}_{n \in \Lambda}$ is noncompact.

Step 2. Given $\lambda \in \mathrm{sp}_c(T_1)$, take the noncompact sequence of unit vectors $\{h_n\}$ such that $\lim_{n \to \infty} (T_1 - \lambda I) h_n = 0$. Let $g = \lim_{n \in \Lambda_1} h_n$ in the weak topology. According to Step 1 we can choose $\{h_n\}_{n \in \Lambda_2}$, $\Lambda_2 \subset \Lambda_1$ with noncompact sequence of differences $\{h_{n+1} - h_n\}_{n \in \Lambda_2}$. The compactness of $T_1 - T_2$ forces convergence in norm

$$\lim_{n \in \Lambda_1} (T_1 - T_2)\, h_n = (T_1 - T_2)\, g, \qquad \lim_{n \in \Lambda_2} (T_1 - T_2)(h_{n+1} - h_n) = 0.$$

Hence

$$\lim_{n \in \Lambda_2} (T_2 - \lambda I)(h_{n+1} - h_n) = 0$$

and $\lambda \in \mathrm{sp}_c(T_2)$ as needed. ∎

We begin the proper business of this section by developing operator theoretic arguments related to orthogonal polynomials on the unit circle (cf. [14, Section 3]).

For orthogonal polynomials on the real line an intimate relationship with infinite Jacobi matrices containing the coefficients of the three-term recurrence relation for the orthonormal polynomials is well known (cf. [1, Chapter 4]). These Jacobi matrices are symmetric tridiagonal matrices which can be extended to self-adjoint operators acting in the Hilbert space ℓ^2.

For orthogonal polynomials on the unit circle there is a similar relationship with infinite matrices, but instead of self-adjoint tridiagonal matrices (for determinate moment problems on the real line) the unitary Hessenberg matrices (for measures outside the Szegő class) come into play.

Fundamental results of N. I. Akhiezer, A. N. Kolmogorov, M. G. Krein, V. I. Smirnov, and G. Szegő imply that given a probability measure μ on $\mathbb{T}$ with infinite support, the system of orthonormal polynomials $\{\varphi_n(\mu)\}_{n=0}^{\infty}$ forms an orthonormal basis in $L^2(\mu, \mathbb{T})$ if and only if

$$\log \mu' \notin L^1(\mathbb{T}) \Leftrightarrow \sum_n |\Phi_n(\mu, 0)|^2 = \infty$$

(cf. [15, Theorem 3.3(a), p. 49]). Throughout the rest of the paper we put $a_0 = 1$,

$$a_n = a_n(\mu) \overset{\text{def}}{=} \Phi_n(\mu, 0), \qquad \rho_n^2 = \rho_n^2(\mu) \overset{\text{def}}{=} 1 - |a_n|^2, \quad n \in \mathbb{N} \overset{\text{def}}{=} \{1, 2, ...\}.$$

In what follows a key role is played by the unitary multiplication operator $U = U(\mu)$ which acts on $L^2(\mu, \mathbb{T})$ by

$$[Uf](t) = tf(t), \qquad t \in \mathbb{T}, \quad f \in L^2(\mu, \mathbb{T}), \tag{5}$$

and its matrix representation $\hat{U} = \hat{U}(\mu)$ in the orthonormal basis $\{\varphi_n(\mu)\}_{n=0}^{\infty}$

$$\hat{U} = \begin{pmatrix} u_{00} & u_{01} & \cdots \\ u_{10} & u_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad u_{ij} = \langle U\varphi_j, \varphi_i \rangle_\mu, \tag{6}$$

where

$$u_{ij} = \begin{cases} -a_{j+1} \overline{a_i} \prod_{k=i+1}^{j} \rho_k, & i < j+1, \\ \rho_{j+1}, & i = j+1, \\ 0, & i > j+1 \end{cases} \tag{7}$$

for $i, j \in \mathbb{Z}^+$ (cf. [14, p. 401]). Infinite matrices such as (6)–(7) in which all entries below the subdiagonal vanish are called (upper) *Hessenberg* matrices.

We can view the infinite matrix (6)–(7) as a unitary operator $\hat{U}$ in ℓ^2 which is unitarily equivalent to the multiplication operator U. In particular, supp(σ) agrees with the spectrum sp$(\hat{U})$.

THEOREM 5. *Let $\mu \in \mathscr{S}$. Denote by $\mathscr{L}$ the set of all limit points of the sequence $\{z_n \stackrel{\text{def}}{=} -a_{n+1}\overline{a_n}\}$. Then*

$$\{\operatorname{supp}(\mu)\}' = \mathscr{L}. \tag{8}$$

Proof. It is clear that $\lim_{n\to\infty} |z_n| = 1$ under assumption (2), so that $\mathscr{L} \subset \mathbb{T}$.

We adopt the argument from [14, Theorem 3]. Let S be the left shift operator in ℓ^2 with the adjoint S^*. Write the operator series expansion for the matrix $\hat{U}$

$$\hat{U} = S^* D_{-1}(\mu) + \sum_{j=0}^{\infty} D_j(\mu)\, S^j, \tag{9}$$

where each $D_j(\mu)$ is a diagonal operator in ℓ^2

$$D_j(\mu) = \operatorname{diag}(D_{j0}, D_{j1}, \ldots),$$
$$D_{ji} \stackrel{\text{def}}{=} \begin{cases} u_{i,\, j+i} = \langle U\varphi_{j+i}, \varphi_i \rangle_\mu, & j \in \mathbb{Z}^+ \\ u_{i+1,\, i} = \langle U\varphi_i, \varphi_{i+1} \rangle_\mu, & j = -1. \end{cases} \tag{10}$$

This infinite series converges in the operator norm. To see this note that $\|S\| = 1$ and $\{\|D_j(\mu)\|\}_{j=-1}^{\infty}$ decreases exponentially in view of (2). What is more to the point, (2) and (10) yield compactness of all diagonal operators D_j, but D_0. Hence we can write $\hat{U} = D_0 + K$ with some compact operator K. By H. Weyl's theorem we have $\operatorname{sp}_c(\hat{U}) = \operatorname{sp}_c(D_0)$. But $\hat{U}$ is unitarily equivalent to the multiplication operator and D_0 is a diagonal operator. The desired result drops out immediately if we take into account Examples 3 and 4. ∎

EXAMPLE 6. Let M be any closed set on $\mathbb{T}$. An example of the measure $\mu \in \mathscr{S}$ with $\{\operatorname{supp}(\mu)\}' = M$ can easily be constructed. Indeed, let $\{\omega_k\}_{k \geq 1}$ be a countable set of points in $[0, 2\pi)$ such that $\{e^{i\omega_k}\}$ is a dense set in $-M$. Let us produce a sequence $\{\xi_k\}$ by the recipe

$$\{\xi_k\} \stackrel{\text{def}}{=} \{\omega_1; \omega_1, \omega_2; \omega_1, \omega_2, \omega_3; \ldots\}.$$

Each point ω_n is clearly a limit point of $\{\xi_k\}$ and hence

$$\{e^{i\xi_k}\}' = -M. \tag{11}$$

Next, define a sequence $\{\vartheta_k\}$ by

$$\vartheta_n \stackrel{\text{def}}{=} \sum_{j=1}^{n} \xi_j, \qquad \vartheta_n - \vartheta_{n-1} = \xi_n.$$

The measure $\mu \in \mathcal{S}$ now comes in as one with reflection coefficients $a_n(\mu) \stackrel{\text{def}}{=} (1-1/n)\, e^{i\vartheta_n}$. In fact,

$$z_n = -a_{n+1}\,\overline{a_n} = -\frac{n-1}{n+1}\, e^{i\xi_{n+1}},$$

so that the set of all limit points of the sequence $\{z_k\}$ is M.

In particular, there are measures $\mu \in \mathcal{S}$ with $\operatorname{supp}(\mu) = \mathbb{T}$.

Remark 7. Let us point out that nothing is claimed about the character of singularity of the measure in Example 6.[1] Following [8, Theorem 6] one can construct a pure point measure μ with given support M, such that

$$\limsup_{n\to\infty} |a_n(\mu)| = 1. \tag{12}$$

3. MEASURES WITH FINITE DERIVED SET OF SUPPORT

Within the framework of operator theoretic considerations in Section 2 the unit circle analogue of Theorem C is obvious.

THEOREM 8. *Let μ be a probability measure on the unit circle with in finite support and U be multiplication operator* (5). *Then the following statements are equivalent.*

(1) $\mu \in \mathcal{K}(\tau_1, ..., \tau_N)$.

(2) *For any polynomial P the operator $P(U)$ is compact in $L^2(\mu, \mathbb{T})$ if and only if $P(\tau_j) = 0$ for $j = 1, 2, ..., N$.*

Proof. Assume that $\{\operatorname{supp}(\mu)\}' = \{\tau_1, \tau_2, ..., \tau_N\}$ holds. By the Spectral Mapping Theorem for every polynomial P with $P(\tau_j) = 0$, $j = 1, 2, ..., N$, the spectrum of $P(U)$ has no nonzero limit points. The operator $P(U)$ being a polynomial of a unitary operator is normal. Since all its eigenvalues are simple, this operator is compact (cf. [16, Problem 133]). On the other hand, if $P(\tau_m) \neq 0$ for some m, then the spectrum of $P(U)$ has a nonzero limit point and hence this operator is *a fortiori* noncompact.

Conversely, put $P_N(z) \stackrel{\text{def}}{=} \prod_{k=1}^{N} (z-\tau_k)$. As $P_N(U)$ is a compact operator, much the same argument as above shows that $\{\operatorname{supp}(\mu)\}' \subset \{\tau_1, \tau_2, ..., \tau_N\}$. If this inclusion were a proper one, then $\tau_m \notin \operatorname{supp}(\mu)$ for some m and $P_{N-1}(U)$ would be a compact operator with $P_{N-1}(z) \stackrel{\text{def}}{=} \prod_{k\neq m} (z-\tau_k)$. The contradiction completes the proof. ∎

[1] The measure is certainly a pure point one whenever $\{\operatorname{supp}(\mu)\}'$ is a finite set.

Let T be a linear operator in a Hilbert space with an orthonormal basis $\{e_n\}_{n=0}^{\infty}$. If T is compact, then $\lim_{n\to\infty}(Te_n, e_{n+k})=0$ for all $k\in\mathbb{Z}$. Moreover, these conditions are equivalent as long as the matrix of T in the basis $\{e_n\}$ is finite–banded (see, e.g. [2, pp. 59–60]). Therefore in the real line setting of Theorem C the operator $p(A)$ is compact whenever

$$\lim_{n\to\infty}\langle p(A)\,p_n, p_{n+k}\rangle_\sigma = \lim_{n\to\infty}\int p(x)\,p_n(x)\,p_{n+k}(x)\,d\sigma = 0, \qquad k\in\mathbb{Z}.$$

The point is that A is represented by a Jacobi matrix in the basis $\{p_n\}$ and thereby the matrix for $p(A)$ is N–banded.

In the unit circle setting the matrix representation for $P(U)$ in the basis $\{\varphi_n\}$ is no longer finite–banded. That is why it is not clear at once whether the conclusion of Theorem 8 holds under a seemingly weaker assumption

$$\lim_{n\to\infty}\int_{\mathbb{T}} P(\zeta)\,\varphi_n(\zeta)\,\overline{\varphi_{n+k}(\zeta)}\,d\mu = \lim_{n\to\infty}\langle P(U)\,\varphi_n, \varphi_{n+k}\rangle_\mu = 0, \qquad k\in\mathbb{Z}. \tag{13}$$

By Theorem A the latter is true for $N=1$.

THEOREM 9. *The operator $P(U)$ is compact in $L^2(\mu,\mathbb{T})$ if and only if* (13) *holds.*

Proof. We only have to prove that $P(U)$ is compact as long as (13) holds. Put $P(z)=\prod_{k=1}^{N}(z-z_k)$ and write infinite series (9) for $P(\hat U)$

$$P(\hat U) = \prod_{k=1}^{N}\left\{S^*\,D_{-1}(\mu) + (D_0 - z_k I) + + \sum_{j=1}^{\infty} D_j(\mu)\,S^j\right\}, \tag{14}$$

which after termwise multiplication gives

$$P_N(\hat U) = \sum_{m=1}^{N}(S^*)^m\,\Delta_{-m}(\mu) + \sum_{j=0}^{\infty}\Delta_j(\mu)\,S^j. \tag{15}$$

The limit relation (13) precisely means that all diagonal operators Δ_l in (15) are compact. The procedure of computing Δ_l explicitly from (14)–(15) does not seem feasible in general for $N\geqslant 3$ (we shall handle the case $N=2$ later on). The only exception is the lowest nonzero diagonal which corresponds to Δ_{-N}. The computation now can be easily carried out. We have from (14)–(15)

$$(S^*)^N\,\Delta_{-N}(\mu) = S^*\,D_{-1}(\mu)\,S^*\,D_{-1}(\mu)\cdots S^*\,D_{-1}(\mu).$$

Applying repeatedly the "commutation" rule

$$\operatorname{diag}(d_0, d_1, ...)\, S^* = S^* \operatorname{diag}(d_1, d_2, ...)$$

leads to the equality

$$\Delta_{-N}(\mu) = \operatorname{diag}(\Delta_{-N,0}, \Delta_{-N,1}, ...), \qquad \Delta_{-N,i} = \prod_{j=1}^{N} \rho_{i+j}.$$

Hence (13) implies

$$\lim_{n\to\infty} \prod_{j=1}^{N} \rho_{n+j} = 0. \tag{16}$$

It follows from (16) and (9) that $D_j(\mu)$ are compact at least for $j \geqslant N$ and the sequence $\{\|D_j(\mu)\|\}_{j=-1}^{\infty}$ decreases exponentially. Therefore the sequence $\{\|\Delta_j(\mu)\|\}_{j=-N}^{\infty}$ decreases exponentially and the series in (14) and (15) converge in the operator norm. Since the set of compact operators in a Hilbert space form a closed ideal, the series (15) produces the compact operator, as needed. ∎

By [20, Lemma 4, p. 110] μ is singular as long as (12) holds. The converse statement is not at all true: there are plenty of singular measures with $\lim_{n\to\infty} a_n = 0$ (cf. [17], [18], [21]). However, the following partial converse is valid.[2]

Corollary 10. *Let* $\mu \in \mathcal{K}(\tau_1, ..., \tau_N)$. *Then* $\limsup_{n\to\infty} |a_n(\mu)| = 1$.

Proof. By the previous theorems, (13) and, in particular, (16) are valid. Hence

$$\liminf_{n\to\infty} \rho_{nN+l} = 0$$

for some $1 \leqslant l \leqslant N$. Note that $\rho_k^2 = 1 - |a_k|^2$. ∎

There is yet another (more natural in a sense) way to treat statements of this kind which rests upon the orthogonality relations and the explicit formula for the leading coefficient $\kappa_n(\mu)$ (cf. [11, p. 7])

$$\int_{\mathbb{T}} P(\zeta)\, \varphi_k(\zeta)\, \overline{\varphi_{k+N}(\zeta)}\, d\mu = \frac{\kappa_k(\mu)}{\kappa_{k+N}(\mu)} = \rho_{k+1} \cdots \rho_{k+N} \tag{17}$$

for an arbitrary monic polynomial P of degree N. We can extend the result obtained in Corollary 10 by assuming the derived set of $\operatorname{supp}(\mu)$ to be

[2] For the real line counterpart see [8, Theorem 4].

infinite with the "high degree of concentration" of the masspoints (cf. [8, Theorem 5] for the real line case).

THEOREM 11. *Let $Q=\{\xi_n\}_{n\geqslant 1}$ be a set of points on $\mathbb{T}$ and $\{N(n)\}$ be a strictly increasing subsequence of positive integers. Let $Q_n=\{\xi_1, ..., \xi_{N(n)}\}$ and denote by $A(\varepsilon_n)$ the ε_n-neighborhood of Q_n. Assume that a measure μ on $\mathbb{T}$ is subject to the following condition: the portion of* supp(μ) *outside $A(\varepsilon_n)$ is finite for each $n\geqslant 1$. Then* (12) *holds provided* $\lim_{n\to\infty} \varepsilon_n^{1/N(n)}=0$.

Proof. Fix n and write (17) for $P(z)=\prod_{j=1}^{N(n)}(z-\zeta_j)$ and $N=N(n)$

$$\rho_{k+1}\cdots\rho_{k+N(n)}=\left\{\int_{A(\varepsilon_n)} P(\zeta)\,\varphi_k(\zeta)\,\overline{\varphi_{k+N(n)}(\zeta)}\,d\mu\right.$$

$$\left.+\int_{B(\varepsilon_n)} P(\zeta)\,\varphi_k(\zeta)\,\overline{\varphi_{k+N(n)}(\zeta)}\,d\mu\right\}$$

where $B(\varepsilon_n)$ is a finite set. Take $k=k(n)$ large enough to have $|\varphi_k(\zeta)|<\varepsilon_n$ for all $\zeta\in B(\varepsilon_n)$. Then

$$\rho_{k+1}\cdots\rho_{k+N(n)}\leqslant 2^{N(n)}\varepsilon_n+2^{N(n)}\varepsilon_n$$

which implies $\liminf \rho_n=0$ as claimed. ∎

It turns out that the computation of $P_N(\hat{U})$ can be performed explicitly for $N=2$ (see [7, p. 103] for the corresponding real line result).

THEOREM 12. *A probability measure $\mu\in\mathcal{K}(\tau_1,\tau_2)$ if and only if*

(i) $\lim_{n\to\infty}\rho_n\rho_{n+1}=0$;

(ii) $\lim_{n\to\infty}\rho_n(a_n\overline{a_{n-1}}+a_{n+1}\overline{a_n}+\tau_1+\tau_2)=0$;

(iii) $\lim_{n\to\infty}((a_{n+1}\overline{a_n}+\tau_1)(a_{n+1}\overline{a_n}+\tau_2)-\rho_{n+1}^2a_{n+2}\overline{a_n}-\rho_n^2a_{n+1}\times\overline{a_{n-1}})=0$.

Proof. Assume first that $\mu\in\mathcal{K}(\tau_1,\tau_2)$. Write

$$P_2(\hat{U})=(\hat{U}-\tau_1)(\hat{U}-\tau_2)=\|w_{kj}\|_0^\infty,$$

$$w_{kj}=\sum_{i=0}^{\infty}(u_{ki}-\tau_1\,\delta_{ki})(u_{ij}-\tau_2\,\delta_{ij}).$$

By Theorem 8 $P_2(\hat{U})$ is compact and hence for all $k\in\mathbb{Z}$

$$\lim_{n\to\infty} w_{n,n+k}=\lim_{n\to\infty}\langle P_2(\hat{U})\,e_{n+k},e_n\rangle=0. \tag{18}$$

It is not hard to see from (7) that (18) with $k=-2,-1,0$ is equivalent to (i), (ii), and (iii), respectively.

Conversely, let (i)–(iii) hold. Much as in Theorem 9 write

$$P_2(\hat{U}) = \prod_{k=1}^{2} \left\{ S^* D_{-1}(\mu) + (D_0 - z_k I) + \sum_{j=1}^{\infty} D_j(\mu) S^j \right\}$$

$$= (S^*)^2 \Delta_{-2}(\mu) + S^* \Delta_{-1}(\mu) + \Delta_0(\mu) \sum_{j=1}^{\infty} \Delta_j(\mu) S^j. \qquad (19)$$

It follows from (i) and (10) that $D_j(\mu)$ are compact for $j \geqslant 2$ and $\{\|D_j(\mu)\|\}_{j=-1}^{\infty}$ decreases exponentially. Therefore the sequence $\{\|\Delta_j(\mu)\|\}_{j=-N}^{\infty}$ decreases exponentially and the series in (19) converge in the operator norm. Now, this is a matter of brute force calculation to verify that (i), (ii), and (iii) provide compactness of Δ_{-2}, $\Delta_{\pm 1}$, and Δ_0, respectively. Thus $P_2(\hat{U})$ is a compact operator. An application of Theorem 8 completes the proof. ∎

Let us illustrate Theorem 12 by two examples.

EXAMPLE 13. We construct a measure μ with almost periodic (with period 2) reflection coefficients $a_n(\mu)$ such that

$$\lim_{n\to\infty} a_{2n+1} = \zeta_1, \qquad |\zeta_1| \leqslant 1, \quad \lim_{n\to\infty} a_{2n} = \zeta_2, \quad |\zeta_2| = 1. \qquad (20)$$

Then

$$\lim_{n\to\infty} a_{2n} \overline{a_{2n-1}} = \zeta_0 \stackrel{\text{def}}{=} \zeta_2 \overline{\zeta_1}, \qquad \lim_{n\to\infty} a_{2n+1} \overline{a_{2n}} = \overline{\zeta_0}. \qquad (21)$$

Now, take two points τ_1, τ_2 on $\mathbb{T}$ which satisfy

$$\tau_1 = \overline{\tau_2}, \qquad \tau_1 + \overline{\tau_1} = -(\zeta_0 + \overline{\zeta_0}). \qquad (22)$$

Conditions (i) and (ii) clearly follow from (21)–(22). To check (iii) note first that by (22)

$$(\zeta_0 + \tau_1)(\zeta_0 + \tau_2) = (\overline{\zeta_0} + \tau_1)(\overline{\zeta_0} + \tau_2)$$

so that

$$\lim_{n\to\infty} (a_{n+1} \overline{a_n} + \tau_1)(a_{n+1} \overline{a_n} + \tau_1) = (\zeta_0 + \tau_1)(\zeta_0 + \tau_2) = 1 - |\zeta_0|^2.$$

Next,

$$\lim_{n\to\infty} \rho_{2n+1}^2 a_{2n+2}\,\overline{a_{2n}} = (1-|\zeta_1|^2)\,|\zeta_2|^2 = 1-|\zeta_1|^2,$$

$$\lim_{n\to\infty} \rho_{2n}^2 a_{2n+1}\,\overline{a_{2n-1}} = (1-|\zeta_2|^2)\,|\zeta_1|^2 = 0.$$

The rest is immediate from $|\zeta_0| = |\zeta_1|$.

EXAMPLE 14. The symmetrized Al-Salam–Carlitz polynomials for the unit circle are introduced in [22, pp. 89–90].

Define a sequence H_n of monic polynomials on the real line by the recurrence (cf. [6, Chapter 6, Section 10])

$$H_{n+1}(x) = xH_n(x) - \lambda_n H_{n-1}(x), \qquad n\in\mathbb{Z}^+, \quad H_{-1}=0, \quad H_0=1 \tag{23}$$

with

$$\lambda_{2n} = 1-q^n, \qquad \lambda_{2n+1} = bq^n, \quad 0<q<1, \quad b>0.$$

The polynomials are known to be orthogonal with respect to a discrete measure σ on the real line, concentrated at two sequences

$$\sigma\{\pm\sqrt{1-q^k}\} = \gamma_k > 0, \qquad \sigma\{\pm\sqrt{1+bq^k}\} = \delta_k > 0, \quad k\in\mathbb{Z}^+.$$

The genuine interval of orthogonality is now $[-\sqrt{1+b}, \sqrt{1+b}]$, so that after the linear change of variables we come to the transformed monic polynomials

$$\hat{H}_n(x) = (1+b)^{-n/2}\, H_n(x\sqrt{1+b})$$

which are orthogonal with respect to a discrete measure $\hat{\sigma}$

$$\hat{\sigma}\left\{\pm\sqrt{\frac{1-q^k}{1+b}}\right\} = \gamma_k, \qquad \hat{\sigma}\left\{\pm\sqrt{\frac{1+bq^k}{1+b}}\right\} = \delta_k,$$

and satisfy the three–term recurrence similar to (23) with $\hat{\lambda}_n = \lambda_n(1+b)^{-1}$. Going over to the unit circle and denoting the associated measure by $\hat{\mu}$, we have by Geronimus' formulas (cf. [12, Theorem 31.1])

$$\Phi_{2n-1}(\hat{\mu}, 0) = 0, \qquad \Phi_{4n}(\hat{\mu}, 0) = 2q^n - 1, \quad \Phi_{4n-2}(\hat{\mu}, 0) = \frac{1-b}{1+b}, \quad n\in\mathbb{N}.$$

The measure $\hat{\mu}$, which can be viewed as a measure on $[-\pi, \pi)$ is symmetric and, when restricted to the half-interval $[0, \pi)$, is concentrated at two sequences $\{\vartheta_k^{\pm}\}$ and $\{\xi_k^{\pm}\}$ defined by

$$\cos \vartheta_k^{\pm} \overset{\text{def}}{=} \pm\sqrt{\frac{1-q^k}{1+b}}, \qquad \cos \xi_k^{\pm} \overset{\text{def}}{=} \pm\sqrt{\frac{1+bq^k}{1+b}}.$$

There are two limit points $\omega^{\pm}$ in $\operatorname{supp} \hat{\mu}$ on $[0, \pi]$

$$\cos \omega^{\pm} \overset{\text{def}}{=} \pm\frac{1}{\sqrt{1+b}}$$

and $\lim_{n\to\infty} \xi_n^{\pm} = \lim_{n\to\infty} \vartheta_n^{\pm} = \omega^{\pm}$. Notice that $\xi_n^{+} + \xi_n^{-} = \vartheta_n^{+} + \vartheta_n^{-} = \pi$. The conditions $\Phi_{2n-1}(\hat{\mu}, 0) = 0$ mean that $\hat{\mu}$ is "sieved".

Now, define a measure μ, which is concentrated at the sequences $\{\pm\tau_k^{\pm}\}$ and $\{\pm\eta_k^{\pm}\}$ with

$$\tau_k^{+} = 2\vartheta_k^{+}, \qquad \tau_k^{-} = 2\vartheta_k^{-} - 2\pi = -2\vartheta_k^{+},$$

$$\eta_k^{+} = 2\xi_k^{+}, \qquad \eta_k^{-} = 2\xi_k^{-} - 2\pi = -2\xi_k^{+},$$

and $\mu\{2x\} = \hat{\mu}\{x\}$. We have

$$\lim_{n\to\infty} \tau_n^{\pm} = \lim_{n\to\infty} \eta_n^{\pm} = \pm 2\omega^{+}$$

and hence the derived set of $\operatorname{supp}(\mu)$ with μ viewed as a measure on $\mathbb{T}$, consists of two points

$$\tau_1 = \overline{\tau_2}, \qquad \tau_1 \overset{\text{def}}{=} \exp\{2i\omega^{+}\}.$$

As for the reflection coefficients they are given by

$$\Phi_{2n}(\mu, 0) = 2q^n - 1, \qquad \Phi_{2n-1}(\mu, 0) = \frac{1-b}{1+b}, \quad n \in \mathbb{N}.$$

We complete the paper with the example of another kind, related to the López classes of measures (cf. [4]).

Definition 15. A probability measure μ with reflection coefficients a_n belongs to the *López* class $\mathscr{L}_N$, $N \in \mathbb{N}$, if

$$\lim_{n\to\infty} |a_{nN+j}| = r_j, \qquad \lim_{n\to\infty} \frac{a_{nN+j+1}}{a_{nN+j}} = \beta_j, \quad j = 1, 2, ..., N. \tag{24}$$

We assume further that $0 < r_j \leqslant 1$ so that β_j are nonzero complex numbers. It is convenient to extend both sequences $\{r_j\}$ and $\{\beta_j\}$ as N-periodic

$$r_{nN+j} \stackrel{\text{def}}{=} r_j, \qquad \beta_{nN+j} \stackrel{\text{def}}{=} \beta_j, \quad n \in \mathbb{N}, \quad j = 1, 2, ..., N,$$

which implies (24) to be in effect for all $j \in \mathbb{N}$. Moreover, applying the second relation in (24) repeatedly yields

$$\lim_{n\to\infty} \frac{a_{nN+j+s}}{a_{nN+j}} = \beta_j \beta_{j+1} \cdots \beta_{j+s-1}, \qquad j, s \in \mathbb{N}. \tag{25}$$

In particular, for $j = 1$, $s = N$ we have

$$\lim_{n\to\infty} \frac{a_{nN+N+1}}{a_{nN+1}} = \beta_1 \beta_2 \cdots \beta_N$$

and hence $|\beta_1 \beta_2 \cdots \beta_N| = 1$. Let us impose the normalization condition (this is just a matter of rotation)

$$\beta_1 \beta_2 \cdots \beta_N = 1, \tag{26}$$

which is equivalent to $\beta_{m+1} \beta_{m+2} \cdots \beta_{m+N} = 1$ for $m \in \mathbb{N}$ due to periodicity of the extension.

Similarly, from (25) with $j = 1$, $s \in \mathbb{N}$ it follows that

$$|\beta_1 \beta_2 \cdots \beta_s| = \frac{r_{s+1}}{r_1}.$$

Let us now define "unperturbed reflection coefficients" $\{\tilde{a}_n\}$ by

$$\tilde{a}_0 \stackrel{\text{def}}{=} 1, \qquad \tilde{a}_q \stackrel{\text{def}}{=} r_1 \beta_1 \cdots \beta_{q-1}, \quad q \in \mathbb{N}.$$

Thanks to normalization (26) the sequence $\{\tilde{a}_n\}_1^\infty$ is N-periodic and $|\tilde{a}_q| = r_q$. In accordance with Favard's theorem it gives rise to some probability measure $\tilde{\mu}$ on $\mathbb{T}$ and the multiplication operator on $L^2(\tilde{\mu}, \mathbb{T})$ which is unitarily equivalent to the operator

$$\tilde{U} = \begin{pmatrix} \tilde{u}_{00} & \tilde{u}_{01} & \cdots \\ \tilde{u}_{10} & \tilde{u}_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{27}$$

where

$$\tilde{u}_{ij} = \begin{cases} -\tilde{a}_{j+1}\,\overline{\tilde{a}_i} \prod_{k=i+1}^{j} \hat{\rho}_k & i < j+1, \\ \hat{\rho}_{j+1}, & i = j+1, \\ 0, & i > j+1, \end{cases} \tag{28}$$

for $i, j = 0, 1, ...,$ as long as $r_j < 1$. But if $r_q = 1$ for some q, no measure appears, yet operator $\tilde{U}$ (27)–(28) makes sence and is a unitary operator in ℓ^2.

Let us evaluate the difference between perturbed operator $\hat{U}$ (6)–(7) associated with $\mu \in \mathscr{L}_N$ and unperturbed one $\tilde{U}$ (27)–(28). We can make use of (9)–(10) for both $\hat{U}$ and $\tilde{U}$, since both $\{\|D_j(\mu)\|\}_{j=-1}^{\infty}$ and $\{\|\tilde{D}_j\|\}_{j=-1}^{\infty}$ decrease exponentially under the assumption $\min r_j > 0$, $j = 1, ..., N$. It is not hard to show that the difference $D_j(\mu) - \tilde{D}_j$ is compact for $\mu \in \mathscr{L}_N$ and $j = -1, 0, ...,$ so that $\hat{U} - \tilde{U}$ is compact as well.

Our particular interest concerns a subclass $\tilde{\mathscr{L}}_N$ of the López class for which $\max r_j = 1$, $j = 1, ..., N$. In this instance it is easily seen directly from (28) that the operator $\tilde{U}$ (which has nothing to do with measures on $\mathbb{T}$) takes the form

$$\tilde{U} = \begin{pmatrix} U_0 & & & \\ & U_1 & & \\ & & U_1 & \\ & & & \ddots \end{pmatrix},$$

where U_0 and U_1 are finite dimensional unitary operators. It is clear now that $\mathrm{sp}_c(\tilde{U})$ is a finite set. By H. Weyl's theorem the same is true for the multiplication operator U in $L^2(\mu, \mathbb{T})$. Hence (see Example 3) the derived set of support of $\mu \in \tilde{\mathscr{L}}_N$ is a finite set.

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